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TRANSIENT AND STEADY-STATE ANALYSIS OF SINGLE SWITCHED CAPACITOR CONVERTER

Abstract—The paper derives closed form expressions for transient and steady-state operation of a DC-DC converter with single switched capacitor. To this end, the result of each switching is considered as a point in the iterative process, and the function between the points is reconstructed. As opposed to the commonly accepted approach, when each of the topologies is approximated by a first order circuit, the proposed analysis is carried out for second order circuits. This allows obtaining the waveform of output voltage ripple and paves the way to more accurate calculation of equivalent resistance. The obtained analytical expressions were verified by simulations and an excellent agreement between the results was found.

Index terms—charge pump, discrete-time systems, equivalent resistance, switched capacitor converter.

Introduction

Switched capacitor converters (SCCs) are favored in some applications due to low EMI and compatibility with IC technology. Over the past few decades, ongoing research has shifted towards sophisticated SCCs with large number of capacitors and advanced control circuits. However, only a few studies use analytical methods. The analysis presented in this paper is based on the method of difference equations (Gardner & Barnes, 1942), (Tsyarkin, 1964), (Derusso, Roy, & Close, 1965), which allows finding the solution for the transient and steady-state operation. Although this method is unified, it was applied previously only to the switched inductor converters (Nerone, 2008), (Shoihet & Slonim, 2010), (Krihely, Slonim, & Ben-Yaakov, 2012). However, some analytical methods for analysis of switched capacitor circuits that bear resemblance of the proposed one, were presented in (Lamantia, Maranesi, & Radrizzani, 1994), (Vitchev, 2006), (Karagozler, Goldstein, & Ricketts, 2012), (Ramezani & Yakovlev, 2013).

Let us consider the SCC shown schematically in Fig. 1(a). It comprises four switches $S_1 \dots S_4$ with on-resistances $r_1 \dots r_4$. The corresponding pair of switches is turned on/off by two non-overlapping clocks φ_1 and φ_2 shown in Fig. 1(b). Thus, during t_1 the capacitor C_1 is charged by V_{in} through S_1, S_3 and then, during t_2 , is discharged to the load through S_2, S_4 . Thus, we have two topologies, which are considered separately in Sections II. Applying to each topology the Kirchhoff's voltage and current laws (KVL and KCL), we write a system of two first-order differential equations. These equations are then solved using the Laplace transform. The solution for the first topology defines the initial conditions for the second one. This enables us to compose a system of two first-order difference equations, which is solved using the Z-transform. Thus, for a given period, n , we know the initial values of state variables and functions according to which these variables change. This is the solution is closed form.

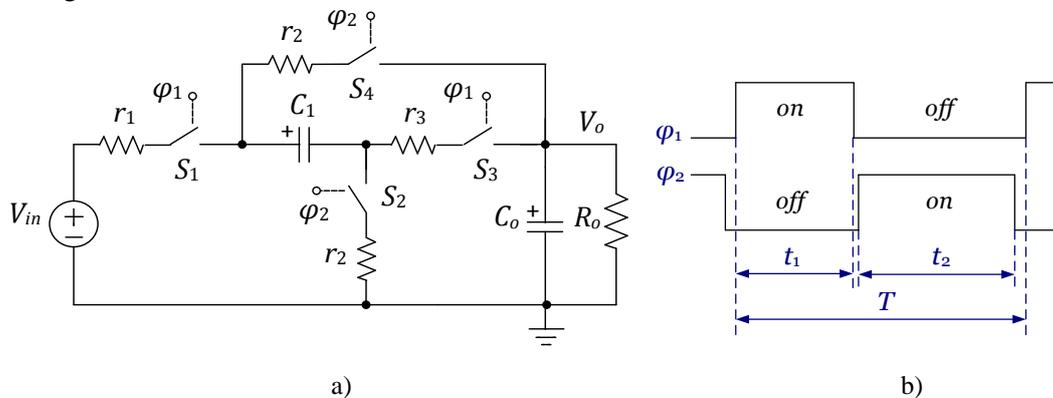


Fig. 1: Considered SCC (a) and two non-overlapping clocks φ_1 and φ_2 (b).

Differential Equations

a) *First topology*

The switches S_1 and S_3 in Fig. 1(a) are turned on during $nT \leq t < nT + t_1$, where n is the number of period. Thus, in the first topology (Fig. 2), $R_1 = r_1 + r_3$.

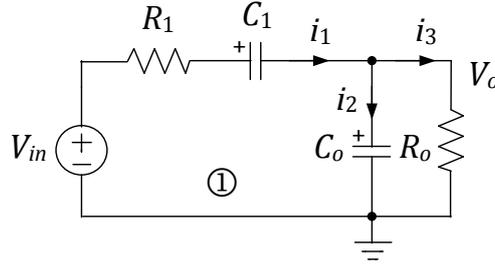


Fig. 2: First topology of the considered SCC.

The KVL and KCL equations for this circuit are:

$$\begin{cases} V_{in} = i_1 R_1 + V_{C1} + V_{Co} \\ i_1 = i_2 + i_3 \end{cases} \quad \text{or} \quad \begin{cases} V_{in} = C_1 \frac{dV_{C1}}{dt} R_1 + V_{C1} + V_{Co} \\ C_1 \frac{dV_{C1}}{dt} = C_o \frac{dV_{Co}}{dt} + \frac{V_{Co}}{R_o} \end{cases} \quad (1)$$

Let us write (1) in the Laplace domain using

$$\mathcal{L}\{K\} = \frac{K}{s} \quad \text{and} \quad \mathcal{L}\left\{\frac{df(x)}{dx}\right\} = sF(s) - f(0) \quad (2)$$

such that

$$\begin{cases} (sR_1 C_1 + 1)V_{C1} + V_{Co} = R_1 C_1 V_{C1}(n) + \frac{V_{in}}{s} \\ -sC_1 V_{C1} + \left(sC_o + \frac{1}{R_o}\right)V_{Co} = C_o V_{Co}(n) - C_1 V_{C1}(n) \end{cases} \quad (3)$$

The solution of (3) can be written as:

$$V_{C1} = \frac{(s - r_1)(s - r_2)}{s(s - s_1)(s - s_2)} V_{C1}(n); \quad V_{Co} = \frac{(s - p)}{(s - s_1)(s - s_2)} V_{Co}(n) \quad (4)$$

where

$$r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \frac{\omega_0^2 V_{in}}{V_{C1}(n)}}; \quad \lambda = \frac{1}{2} \left[\left(\frac{1}{R_1} + \frac{1}{R_o} \right) \frac{1}{C_o} + \frac{V_{in} - V_{Co}(n)}{R_1 C_1 V_{C1}(n)} \right]; \quad (5)$$

$$p = - \left(\frac{V_{in} - V_{C1}(n)}{R_1 C_o V_{Co}(n)} + \frac{1}{R_1 C_1} \right)$$

and

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}; \quad \alpha = \frac{1}{2} \left[\left(\frac{1}{C_1} + \frac{1}{C_o} \right) \frac{1}{R_1} + \frac{1}{R_o C_o} \right]; \quad \omega_0^2 = \frac{1}{R_1 C_1 R_o C_o} \quad (6)$$

The inverse Laplace transform of (4) is:

$$\begin{aligned} \mathcal{L}^{-1}\{V_{C1}\} &= \left(\frac{(r_1 - s_1)(r_2 - s_1)}{s_1(s_1 - s_2)} e^{s_1 t} - \frac{(r_1 - s_2)(r_2 - s_2)}{s_2(s_1 - s_2)} e^{s_2 t} + \frac{r_1 r_2}{s_1 s_2} \right) V_{C1}(n) \\ \mathcal{L}^{-1}\{V_{Co}\} &= \frac{(s_2 - p)e^{s_2 t} - (s_1 - p)e^{s_1 t}}{s_2 - s_1} V_{Co}(n) \end{aligned} \quad (7)$$

Using $t = t_1$ and

$$r_1 r_2 = \frac{\omega_0^2 V_{in}}{V_{C1}(n)}; \quad s_1 s_2 = \omega_0^2; \quad r_1 + r_2 = -2\lambda \quad (8)$$

we can rewrite (7) as:

$$\begin{aligned} V_{C_1}(t_1) &= a + b(V_{in} - V_{C_o}(n)) + cV_{C_1}(n) \\ V_{C_o}(t_1) &= d(V_{in} - V_{C_1}(n)) + eV_{C_o}(n) \end{aligned} \quad (9)$$

where

$$\begin{aligned} a &= \left(1 + \frac{s_1 e^{s_2 t_1} - s_2 e^{s_1 t_1}}{s_2 - s_1}\right) V_{in}; \quad b = \frac{e^{s_2 t_1} - e^{s_1 t_1}}{s_2 - s_1} \left(\frac{1}{R_1 C_1}\right); \\ c &= \frac{\left[s_2 + \left(\frac{1}{R_1} + \frac{1}{R_o}\right) \frac{1}{C_o}\right] e^{s_2 t_1} - \left[s_1 + \left(\frac{1}{R_1} + \frac{1}{R_o}\right) \frac{1}{C_o}\right] e^{s_1 t_1}}{s_2 - s_1}; \\ d &= \frac{e^{s_2 t_1} - e^{s_1 t_1}}{s_2 - s_1} \left(\frac{1}{R_1 C_o}\right); \quad e = \frac{\left(s_2 + \frac{1}{R_1 C_1}\right) e^{s_2 t_1} - \left(s_1 + \frac{1}{R_1 C_1}\right) e^{s_1 t_1}}{s_2 - s_1} \end{aligned} \quad (10)$$

b) Second topology

The switches S_2 and S_4 in Fig. 1(a) are turned on during $nT + t_1 \leq t < (n+1)T$, such that in the second topology (Fig. 3) $R_2 = r_2 + r_4$.

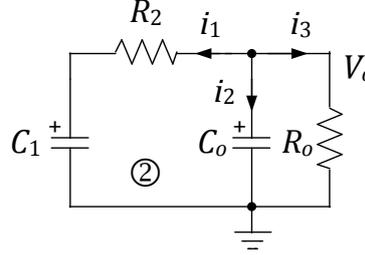


Fig. 3: Second topology of the considered SCC.

The KVL and KCL equations for this circuit are:

$$\begin{cases} V_{C_1} = -i_1 R_2 + V_{C_o} \\ -i_1 = i_2 + i_3 \end{cases} \quad \text{or} \quad \begin{cases} V_{C_1} = -C_1 \frac{dV_{C_1}}{dt} R_2 + V_{C_o} \\ -C_1 \frac{dV_{C_1}}{dt} = C_o \frac{dV_{C_o}}{dt} + \frac{V_{C_o}}{R_o} \end{cases} \quad (11)$$

Using (2), we write (11) in the Laplace domain:

$$\begin{cases} (sR_2 C_1 + 1)V_{C_1} - V_{C_o} = R_2 C_1 V_{C_1}(n) \\ sC_1 V_{C_1} + \left(sC_o + \frac{1}{R_o}\right) V_{C_o} = C_1 V_{C_1}(n) + C_o V_{C_o}(n) \end{cases} \quad (12)$$

The solution of (12) is:

$$V_{C_1} = \frac{(s - p_1)}{(s - s_1)(s - s_2)} V_{C_1}(n); \quad V_{C_o} = \frac{(s - p_2)}{(s - s_1)(s - s_2)} V_{C_o}(n) \quad (13)$$

where

$$p_1 = -\left(\frac{V_{C_1}(n)}{C_o V_{C_o}(n)} + \frac{1}{C_1}\right) \frac{1}{R_2}; \quad p_2 = -\left(\frac{1}{R_o} + \frac{1}{R_2}\right) \frac{1}{C_o} - \left(\frac{1}{R_2 C_1}\right) \frac{V_{C_o}(n)}{V_{C_1}(n)} \quad (14)$$

and

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}; \quad \alpha = \frac{1}{2} \left[\left(\frac{1}{C_1} + \frac{1}{C_o}\right) \frac{1}{R_2} + \frac{1}{R_o C_o} \right]; \quad \omega_0^2 = \frac{1}{R_2 C_1 R_o C_o} \quad (15)$$

The inverse Laplace transform of (13) is:

$$\begin{aligned} \mathcal{L}^{-1}\{V_{C_1}\} &= \frac{(s_2 - p_1)e^{s_2 t} - (s_1 - p_1)e^{s_1 t}}{s_2 - s_1} V_{C_1}(n) \\ \mathcal{L}^{-1}\{V_{C_o}\} &= \frac{(s_2 - p_2)e^{s_2 t} - (s_1 - p_2)e^{s_1 t}}{s_2 - s_1} V_{C_o}(n) \end{aligned} \quad (16)$$

Since for the second topology $t_1 \leq t \leq t_2$, the initial conditions will be $V_{C_1}(n) = V_{C_1}(t_1)$ and $V_{C_o}(n) = V_{C_o}(t_1)$, whereas $t = t_2 - t_1$.

Difference Equations

Substituting (14) and (15) into (16), we obtain the recurrent equations:

$$V_{C_1}(n+1) = fV_{C_1}(t_1) + gV_{C_o}(t_1); \quad V_{C_o}(n+1) = hV_{C_1}(t_1) + kV_{C_o}(t_1) \quad (17)$$

where

$$f = \frac{\left(s_2 + \left(\frac{1}{R_o} + \frac{1}{R_2}\right)\frac{1}{C_o}\right)e^{s_2(t_2-t_1)} - \left(s_1 + \left(\frac{1}{R_o} + \frac{1}{R_2}\right)\frac{1}{C_o}\right)e^{s_1(t_2-t_1)}}{s_2 - s_1};$$

$$g = \frac{e^{s_2(t_2-t_1)} - e^{s_1(t_2-t_1)}}{s_2 - s_1} \left(\frac{1}{R_2 C_1}\right); \quad h = \frac{e^{s_2(t_2-t_1)} - e^{s_1(t_2-t_1)}}{s_2 - s_1} \left(\frac{1}{R_2 C_o}\right); \quad (18)$$

$$k = \frac{\left(s_2 + \frac{1}{R_2 C_1}\right)e^{s_2(t_2-t_1)} - \left(s_1 + \frac{1}{R_2 C_1}\right)e^{s_1(t_2-t_1)}}{s_2 - s_1}$$

Now, substituting (9) into (17), we obtain the difference equations:

$$\begin{cases} V_{C_1}(n+1) = A + BV_{C_o}(n) + GV_{C_1}(n) \\ V_{C_o}(n+1) = E + FV_{C_o}(n) + HV_{C_1}(n) \end{cases} \quad (19)$$

where

$$\begin{aligned} A &= (fb + gd)V_{in} + fa; \quad B = ge - fb; \quad G = fc - gd \\ E &= (hb + kd)V_{in} + ha; \quad F = ke - hb; \quad H = hc - kd \end{aligned} \quad (20)$$

Before we proceed to the solution of (19), let us consider the steady-state operation, where

$$V_{C_1}(n+1) = V_{C_1}(n); \quad V_{C_o}(n+1) = V_{C_o}(n) \quad (21)$$

substituting (21) into (19), we have

$$\begin{cases} (G-1)V_{C_1}(n) + BV_{C_o}(n) = -A \\ HV_{C_1}(n) + (F-1)V_{C_o}(n) = -E \end{cases} \quad (22)$$

The solution of (22) is:

$$V_{C_1}(n) = \frac{BE - A(F-1)}{(F-1)(G-1) - BH}; \quad V_{C_o}(n) = \frac{HA - E(G-1)}{(F-1)(G-1) - BH} \quad (23)$$

Let us write (19) in the Z-domain using

$$Z\{K\} = K \frac{z}{z-1} \quad \text{and} \quad Z\{f(n+1)\} = zF(z) - f(0) \quad (24)$$

such that

$$\begin{cases} (z-G)V_{C_1}(z) - BV_{C_o}(z) = A \frac{z}{z-1} + zV_{C_1}(0) \\ -HV_{C_1}(z) + (z-F)V_{C_o}(z) = E \frac{z}{z-1} + zV_{C_o}(0) \end{cases} \quad (25)$$

The solution of (25) is:

$$V_{C_1}(z) = \frac{(z-F)V_{C_1}(0) + BV_{C_o}(0)}{(z-z_1)(z-z_2)} z + \frac{(z-F)A + BE}{(z-z_1)(z-z_2)} \left(\frac{z}{z-1}\right)$$

$$V_{C_o}(z) = \frac{HV_{C_1}(0) + (z-G)V_{C_o}(0)}{(z-z_1)(z-z_2)} z + \frac{HA + (z-G)E}{(z-z_1)(z-z_2)} \left(\frac{z}{z-1}\right) \quad (26)$$

where

$$z_{1,2} = \frac{G+F \pm \sqrt{(G+F)^2 - 4(GF - BH)}}{2} \quad (27)$$

Since (26) is represented in the form of partial fractions, we can apply the inverse Z-transform to each term separately:

$$\begin{aligned}
 V_{C_1}(n) &= \frac{A(z_2 - F) + BE}{z_2 - z_1} \left(\frac{z_2^n - 1}{z_2 - 1} \right) - \frac{A(z_1 - F) + BE}{z_2 - z_1} \left(\frac{z_1^n - 1}{z_1 - 1} \right) + \\
 &\quad \frac{(z_2 - F)z_2^n - (z_1 - F)z_1^n}{z_2 - z_1} V_{C_1}(0) + \frac{z_2^n - z_1^n}{z_2 - z_1} B V_{C_o}(0) \\
 V_{C_o}(n) &= \frac{E(z_2 - G) + AH}{z_2 - z_1} \left(\frac{z_2^n - 1}{z_2 - 1} \right) - \frac{E(z_1 - G) + AH}{z_2 - z_1} \left(\frac{z_1^n - 1}{z_1 - 1} \right) + \\
 &\quad \frac{(z_2 - G)z_2^n - (z_1 - G)z_1^n}{z_2 - z_1} V_{C_o}(0) + \frac{z_2^n - z_1^n}{z_2 - z_1} H V_{C_1}(0)
 \end{aligned} \tag{28}$$

Simulation Results

To verify the obtained analytical expressions (9), (17) and (28), the circuit shown in Fig. 4 was simulated in PSIM 9.0. Since the PSIM bidirectional switches have zero on-resistance, two external resistors corresponding to R_1 and R_2 in Fig. 2 and Fig. 3 were added. The parameters of the circuit in Fig. 4 are as follows: $V_{in} = 10V$, $V_{C_1}(0) = V_{C_o}(0) = 0V$, $t_1 = t_2 = 5\mu s$, $R_1 = R_2 = 1\Omega$, $C_1 = 10\mu F$, $R_o = 100\Omega$ and $C_o = 100\mu F$.

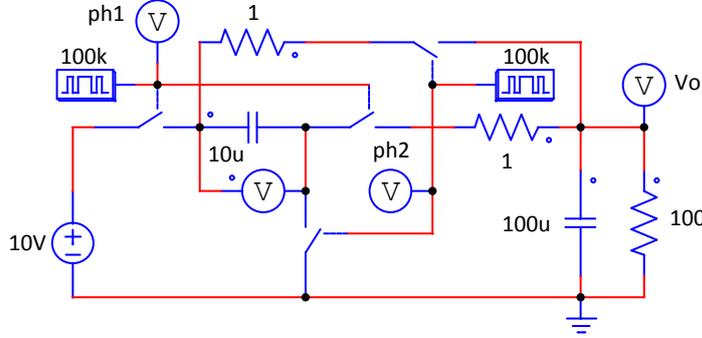
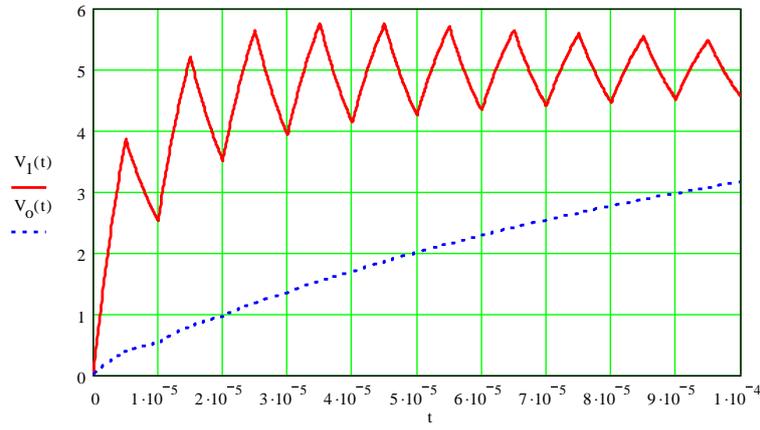


Fig. 4: Simulation circuit for the voltage-halving SCC.

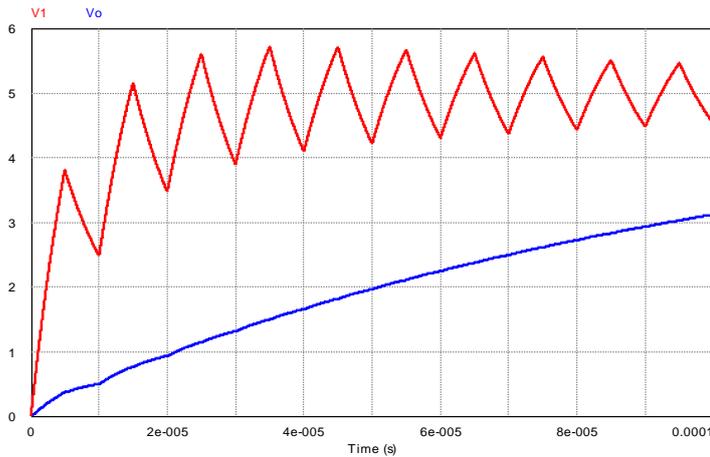
Since $R_1 = R_2$ and $t_1 = t_2$, the expressions (10) and (18) are reduced to:

$$\begin{aligned}
 a &= \left(1 + \frac{s_1 e^{s_2 t_1} - s_2 e^{s_1 t_1}}{s_2 - s_1} \right) V_{in}; \quad b = g = \frac{e^{s_2 t_1} - e^{s_1 t_1}}{s_2 - s_1} \left(\frac{1}{R_1 C_1} \right); \\
 c = f &= \frac{\left[s_2 + \left(\frac{1}{R_1} + \frac{1}{R_o} \right) \frac{1}{C_o} \right] e^{s_2 t_1} - \left[s_1 + \left(\frac{1}{R_1} + \frac{1}{R_o} \right) \frac{1}{C_o} \right] e^{s_1 t_1}}{s_2 - s_1}; \\
 e = k &= \frac{\left(s_2 + \frac{1}{R_1 C_1} \right) e^{s_2 t_1} - \left(s_1 + \frac{1}{R_1 C_1} \right) e^{s_1 t_1}}{s_2 - s_1}; \quad d = h = b \frac{C_1}{C_o}
 \end{aligned} \tag{29}$$

These constants are substituted into (20) and then into (28), which sets the initial conditions for (9) and (17). The voltages across C_1 and C_o during the first ten periods ($0 \leq t \leq 10T$) is shown Fig. 5, which compares the MathCAD calculation and PSIM simulation.



a)



b)

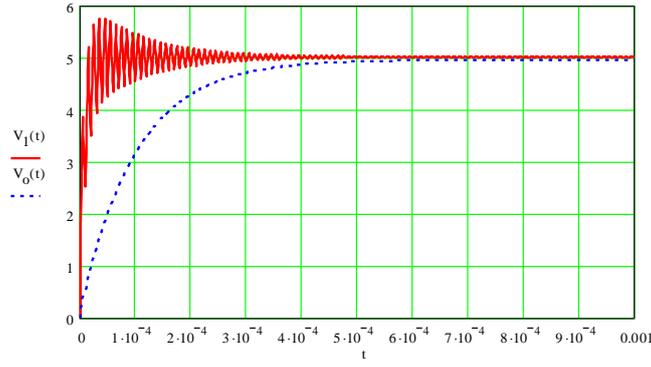
Fig. 5: Voltages across C_1 and C_o for $0 \leq t \leq 10T$, MathCAD (a) and PSIM (b).
The horizontal scale is $10\mu s/div$, i.e. each division is T .

The voltages in Fig. 5 were measured at the points nT and are given in Table I along with the relative error, ε .

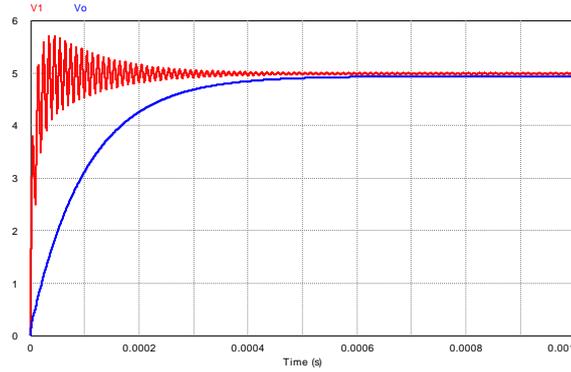
Table I: Measured values of the voltages in Fig. 5.

n	$V_{C1}(n)$		$\varepsilon[\%]$	$V_{Co}(n)$		$\varepsilon[\%]$
	Calc.	Simul.		Calc.	Simul.	
1	2.5146	2.5046	0.398	0.5174	0.5119	1.074
2	3.4986	3.4937	0.140	0.9542	0.9449	0.984
3	3.9147	3.9144	0.008	1.3381	1.3259	0.920
4	4.1172	4.1193	0.051	1.6815	1.6670	0.870
5	4.2366	4.2397	0.073	1.9909	1.9747	0.820
6	4.3211	4.3254	0.010	2.2706	2.2531	0.777
7	4.3890	4.3943	0.121	2.5236	2.5054	0.726
8	4.4474	4.4534	0.135	2.7526	2.7339	0.684
9	4.4991	4.5055	0.142	2.9601	2.9410	0.649
10	4.5454	4.5455	0.002	7.1478	7.1282	0.627

Fig. 6 compares the MathCAD calculation and PSIM simulation for $0 \leq t \leq 100T$. Note that at $t \approx 60T$ the SCC reaches the steady-state.



a)



b)

Fig. 6: Voltages across C_1 and C_o for $0 \leq t \leq 100T$, MathCAD (a) and PSIM (b). The horizontal scale is $100\mu s/div$, i.e. each division is $10T$.

The steady-state voltages are shown Fig. 7. Their discrete values were calculated by (23) and then substituted as the initial conditions into (9) and (17).

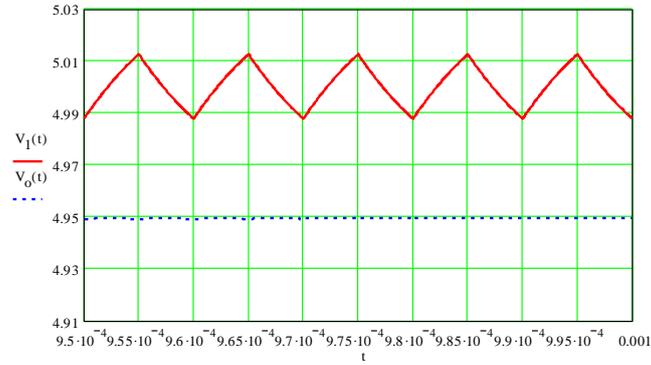


Fig. 7: Steady-state voltages across C_1 and C_o for $95T \leq t \leq 100T$.

Conclusion

Based on the method of difference equations the closed form expressions for the voltages across the capacitors in the voltage-halving SCC were derived. The solution of these equations allows us to predict the SCC behavior in both the transient and steady-state operation. That is for a given period, n , we know the initial values of the voltages and functions according to which these voltages change. The obtained expressions were verified by simulations. As evident from Table I, the deviation between the theoretical and simulation results does not exceed 1.1%. The used method however, is very complex even in the case of the two-phase SCC and its extension to the multi-phase SCC will apparently require some special assumptions.

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